Analysis of Deep Bed Filtration Data: Modeling as a Birth-Death Process

To simulate the performance of a deep bed filter in terms of the pressure drop dynamics under a constant flow condition, a fairly general stochastic model, namely, the birth-death process, which takes into account both blockage of the pores by suspended particles and scouring of deposited particles, is combined with the Carman-Kozeny equation. This model is relatively simple in that the entire bed is spatially lumped and it contains only two parameters, α and β, which are fairly easy to identify. In spite of this simplicity, the model is capable of representing the majority of the available experimental data.

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SCOPE

The use of stochastic models to simulate deep bed filtration has been proposed by J. Litwiniszyn (1963, 1966, 1967, 1968a,b, 1969). These stochastic models can be viewed as the intermediate between the phenomenological models (Iwasaki, 1937; Ives, 1960, 1961; Camp, 1964), in which the form of model equations is assumed from prior knowledge of the phenomena, and the trajectory analysis models (O'Melia and Stum, 1967; Yao et al., 1971; Payatakes, 1973; Rajagopalan and Tien, 1976, 1979; Pendse et al., 1978), in which the trajectories of the particles are painstakingly determined from the force balance equations. The filtration process is complicated and chaotic in nature, and thus the stochastic models derived through probability considerations often generate parameters that are easy to identify and adequately describe the behavior of the filter.

Litwiniszyn (1963) has used a pure birth process where the number of blocked pores in a unit volume of the bed is considered as the model variable. Hsu (1981), and Hsu and Fan (1984) have employed the pure birth process to model the pressure drop across the whole bed; in their model the number of blocked pores is related to the pressure drop by assuming that the scouring of deposited particles is negligible.

One group of research workers (Mintz, 1951; Mintz et al., 1967) have considered that deposits accumulated in the filter medium have a structure with uneven strength. Under the action of hydrodynamic forces due to the flow of water through the medium, which increase with an increase in the pressure drop, this structure is partially destroyed; a portion of the particles less strongly linked to the other particles is detached from the grains. Based on this observation, Litwiniszyn (1966) has developed a birth-death model. In the present work, this birth-death model, which takes into account both blockage of the pores and scouring of deposited particles, is combined with the Carman-Kozeny equation for the pressure drop through the packed bed to describe its temporal change during a filtration run. The experimental data for a wide variety of sources have been analyzed in the light of the present model.

It should be emphasized that in the stochastic modeling of deep bed filtration, the concept of a pore refers to a finite but small space within the filter bed which is susceptible to the deposition of solid particles. Each pore may or may not have a clearly defined boundary, and a space surrounded by several grains or particles may be composed of more than one pore. Once occupied by the particles deposited from the flow, thus preventing additional suspension to pass through it, the pore is said to be blocked, without regard to the nature of deposition.

CONCLUSIONS AND SIGNIFICANCE

To understand the characteristics of the deep bed filtration process, numerous investigators have spent enormous effort in studying different types of deep beds under wide ranges of operating conditions. It appears that so far little has been done to analyze and mechanistically model their results in a unified manner. A fairly general stochastic model, the birth-death model, which takes into account both blockage of the pores by suspended particles and scouring of deposited particles, has been proposed here to simulate the performance of this process in terms of the pressure drop dynamics under the constant flow condition.

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The model is relatively simple in that the entire bed is spatially lumped, and it contains only two parameters, α and β, the fomer being the blockage constant and the latter the scouring constant. In spite of this simplicity, it has been shown that the model is capable of representing the majority of the available experimental data. This implies that the present model can be utilized to analyze and simulate mechanically the performance of deep bed filters of different types. Furthermore, for scale-up and design, α and β can be correlated with various factors influencing the performance of the filter, such as the properties and concentrations of suspension.

MODEL

A stochastic birth-death model was first employed by Litwiniszyn (1966) to describe pore blockage and scouring in a filtration process. By incorporating the Carman-Kozeny equation, this model has been developed to simulate the pressure drop dynamics through the deep bed filter under the constant flow condition.

Birth-Death Process

A stochastic process $\{N(t)\}$ is a family of random variables describing an empirical process whose development is governed by probabilistic laws. In considering the filtration process in a deep packed bed as a stochastic process, the number of blocked pores in a unit volume of the bed at time t, $N(t)$, can be taken as the random variable (Litwiniszyn, 1966); a specific value of $N(t)$ will be denoted by n. Given $N(t) = n$ it is assumed that for the birth-death process (Chiang, 1980),

1. The conditional probability that a birth event will occur, i.e., that an open pore will be blocked during the time interval $(t,t + \Delta t)$, is $\lambda_n \Delta t + o(\Delta t)$, where λ_n is a function of n.
2. The conditional probability that a death event will occur, i.e., that a blocked pore will be scoured during the time interval $(t,t + \Delta t)$, is $\mu_n \Delta t + o(\Delta t)$, where μ_n is a function of n.
3. The conditional probability that more than one event will occur in this time interval is $o(\Delta t)$, where $o(\Delta t)$ signifies that

$$\lim_{\Delta t \to 0} \frac{o(\Delta t)}{\Delta t} = 0$$

Obviously the probability of no change in the time interval $(t,t + \Delta t)$ is $[1 - \lambda_n \Delta t - \mu_n \Delta t - o(\Delta t)]$. The probability that exactly n pores are blocked at the moment t will be denoted as $p_n(t) = Pr[N(t) = n]$; $n = 0,1,\ldots$ For the two adjacent intervals $(0,t)$ and $(t,t + \Delta t)$, the occurrence of exactly n pores being blocked during the time interval $(0,t + \Delta t)$ can be realized in the following mutually exclusive ways.

1. All n pores are blocked in $(0,t)$, and none in $(t,t + \Delta t)$, with probability $p_n(t)[1 - \lambda_n \Delta t - \mu_n \Delta t - o(\Delta t)]$.
2. Exactly $(n - 1)$ pores are blocked in $(0,t)$, and one pore is blocked in $(t,t + \Delta t)$, with probability $p_{n-1}(t)[\lambda_{n-1}\Delta t + o(\Delta t)]$.
3. Exactly $(n + 1)$ pores are blocked in $(0,t)$, and one pore is scoured in $(t,t + \Delta t)$, with probability $p_{n+1}(t)[\mu_{n+1}\Delta t + o(\Delta t)]$.
4. Exactly $(n - j)$ pores are blocked in $(0,t)$, and j pores are blocked in $(t,t + \Delta t)$, where $2 \le j \le n$, with probability $o(\Delta t)$.
5. Exactly $(n + j)$ pores are blocked in $(0,t)$, and j pores are scoured in $(t,t + \Delta t)$, where $2 \le j \le (n_0 - n)$, with probability $o(\Delta t)$. Considering all these possibilities and combining all quantities of $o(\Delta t)$, gives

$$p_n(t + \Delta t) = p_n(t)[1 - \lambda_n \Delta t - \mu_n \Delta t] + p_{n-1}(t)\lambda_{n-1}\Delta t + p_{n+1}(t)\mu_{n+1}\Delta t + o(\Delta t), \quad n \ge 1 \tag{1}$$

and

$$p_0(t + \Delta t) = p_0(t)[1 - \lambda_0 \Delta t] + p_1(t)\mu_1 \Delta t + o(\Delta t) \tag{2}$$

Rearranging these equations and taking the limit as $\Delta t \to 0$ yields the following master equations (see Appendix A)

$$\frac{dp_n(t)}{dt} = \lambda_{n-1}p_{n-1}(t) - (\lambda_n + \mu_n)p_n(t) + \mu_{n+1}p_{n+1}(t), \quad n \ge 1 \tag{3}$$

and

$$\frac{dp_0(t)}{dt} = -\lambda_0 p_0(t) + \mu_1 p_1(t) \tag{4}$$

Litwiniszyn (1966) has assumed that the intensities of transition, λ_n and μ_n, take the form

$$\lambda_n = \alpha(n_0 - n), \quad n = 0,1,2,\ldots,n_0 \tag{5}$$

$$\mu_n = \beta n \tag{6}$$

where n_0 is the total number of open pores susceptible to blockage at the moment $t = 0$, and α and β are the proportionality constants. The constants α and β may be called the blockage constant and the scouring constant, respectively. Equation 5 implies that the rate of pore blockage is proportional to the number of open pores, and Eq. 6 implies that the rate of pore scouring is proportional to the number of blocked pores. Introducing Eqs. 5 and 6 into Eqs. 3 and 4, respectively, the following equations are obtained:

$$\frac{dp_n(t)}{dt} = \alpha[n_0 - (n - 1)]p_{n-1}(t) - [\alpha(n_0 - n) + \beta n]p_n(t) + \beta(n + 1)p_{n+1}(t), \quad n \ge 1 \tag{7}$$

and

$$\frac{dp_0(t)}{dt} = -\alpha n_0 p_0(t) + \beta p_1(t) \tag{8}$$

At the start of the filtration process all pores are open; thus the initial conditions to Eqs. 7 and 8 may be expressed as

$$p_n(0) = 0 \quad n = 1,2,\ldots,n_0$$

$$p_0(0) = 1 \tag{9}$$

Solution of Eqs. 7 and 8, subject to initial conditions, Eq. 9, yields the distribution of the probabilities. However, such information has limited physical significance. Instead, the mean number of blocked pores at a given moment t, namely, $E[N(t)]$, is determined; it is defined as

$$E[N(t)] = \sum_{n=0}^{n_0} np_n(t) \tag{10}$$

To evaluate this value, it is unnecessary to solve Eqs. 7 and 8. We can use the method of probability generating function defined as

$$G(s,t) = \sum_{n=0}^{n_0} s^n p_n(t) \tag{11}$$

Multiplying both sides of Eq. 7 by the respective s^n's and Eq. 8 by s^0, and summing all resultant equations, the following expression is obtained upon rearrangement (Chiang, 1980; also see Appendix B).

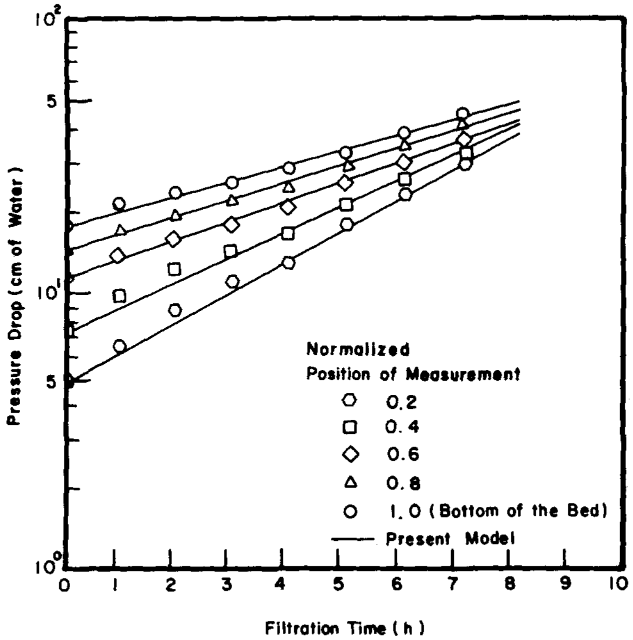

Figure 1. Fitting present model to Deb's (1969) data. Sand size, 0.0647 cm; bed depth, 61 cm; flow rate, 4.9 m/h; Fuller's earth suspension, 45 × 10^{-4} vol %.

Run I: Position 0.2 (α = 0.261 h^{-1}, β = 0.000 h^{-1}, R^2 = 0.993)
Position 0.4 (α = 0.217 h^{-1}, β = 0.000 h^{-1}, R^2 = 0.992)
Position 0.6 (α = 0.164 h^{-1}, β = 0.000 h^{-1}, R^2 = 0.990)
Position 0.8 (α = 0.146 h^{-1}, β = 0.000 h^{-1}, R^2 = 0.986)
Position 1.0 (α = 0.130 h^{-1}, β = 0.000 h^{-1}, R^2 = 0.987)

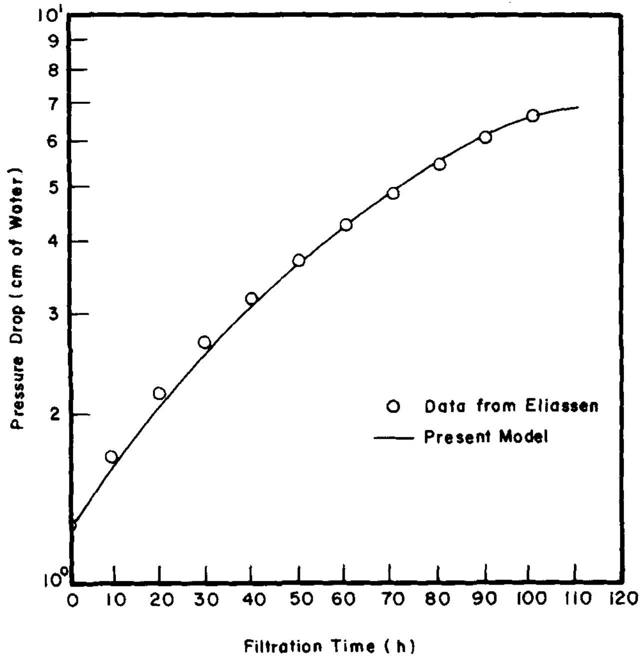

Figure 2. Fitting present model to Eliassen's (1935) data. Sand size, 0.051 cm; bed depth, 61 cm; flow rate, 4.7 m/h; hydrous ferric oxide floc suspension, size, 0.00062 cm, concentration, 50 × 10^{-4} vol %.

Run 6: α = 0.0252 h^{-1}, β = 0.0045 h^{-1}, R^2 = 0.996.

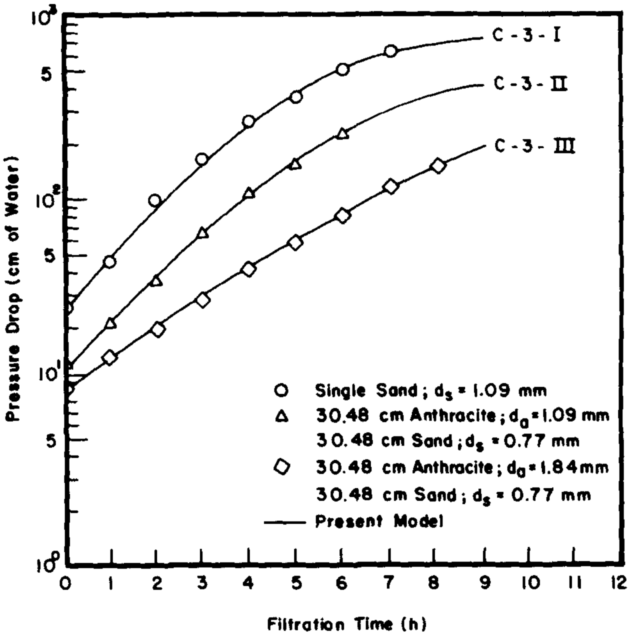

Figure 3. Fitting present model to Huang's (1972) data. Bed depth, 60.96 cm; flow rate, 9.8 m/h; waste water suspension, 12.5 mg/L.

Runs: C-3-I (α = 0.645 h^{-1}, β = 0.0209 h^{-1}, R^2 = 0.995)
C-3-II (α = 0.578 h^{-1}, β = 0.0146 h^{-1}, R^2 = 0.997)
C-3-III (α = 0.382 h^{-1}, β = 0.0073 h^{-1}, R^2 = 0.998)

$$\frac{\partial G(s,t)}{\partial t} = \frac{\partial G(s,t)}{\partial s}[\beta + (\alpha - \beta)s - \alpha s^2] + G(s,t)[\alpha n_0(s-1)] \quad (12)$$

The corresponding initial and boundary conditions are

$$G(s,0) = 1 \quad (13)$$

$$G(1,t) = 1 \quad (14)$$

The solution of Eq. 12 is

$$G(s,t) = \left[\frac{(\alpha s + \beta) - \alpha(s-1)e^{-(\alpha+\beta)t}}{\alpha + \beta}\right]^{n_0} \quad (15)$$

Examination of Eqs. 10 and 11 shows that

$$E[N(t)] = \left.\frac{\partial G(s,t)}{\partial s}\right|_{s=1} \quad (16)$$

Substitution of Eq. 15 into Eq. 16 gives

$$E[N(t)] = \alpha n_0 \left[\frac{1 - e^{-(\alpha+\beta)t}}{\alpha + \beta}\right] \quad (17)$$

The variance of the random variable, Var[$N(t)$], takes the form of

$$\mathrm{Var}[N(t)] = \left.\left\{\frac{\partial^2 G(s,t)}{\partial s^2} + \frac{\partial G(s,t)}{\partial s} - \left[\frac{\partial G(s,t)}{\partial s}\right]^2\right\}\right|_{s=1} = \frac{\alpha n_0}{\alpha + \beta}[1 - e^{-(\alpha+\beta)t}]\left\{1 - \frac{\alpha[1 - e^{-(\alpha+\beta)t}]}{\alpha + \beta}\right\} \quad (18)$$

Constant Flow Filtration

To utilize the results obtained in the preceding subsection, it is necessary to relate the mean value of the number of blocked pores to measurable variables. For the case of constant flow filtration, it is assumed that all pores have the same radius, r. Denoting the

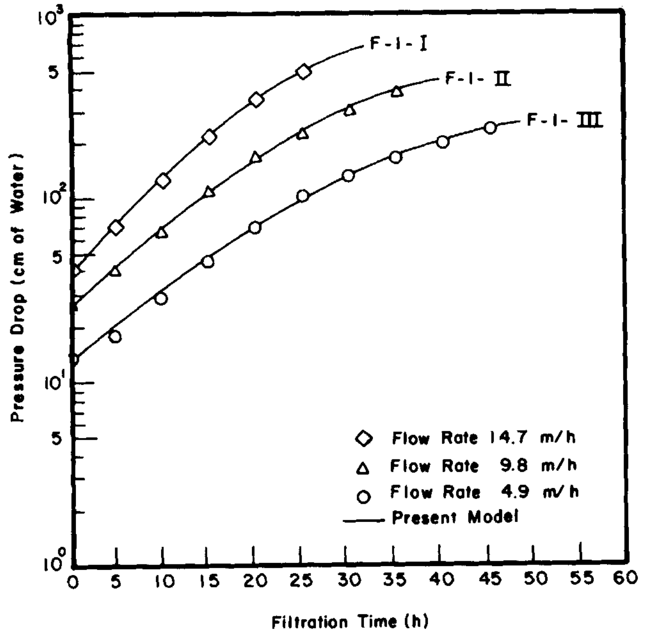

Figure 4. Fitting present model to Huang's (1972) data. Bed depth, 60.96 cm: anthracite, 30.48 cm, d_a = 0.184 cm; sand, 30.48 cm, d_s = 0.055 cm; waste water suspension 12.5 mg/L.

Runs: F-1-I (α = 0.122 h^{-1}, β = 0.0054 h^{-1}, R^2 = 0.999)
F-1-II (α = 0.100 h^{-1}, β = 0.0045 h^{-1}, R^2 = 0.992)
F-1-III (α = 0.089 h^{-1}, β = 0.0040 h^{-1}, R^2 = 0.998)

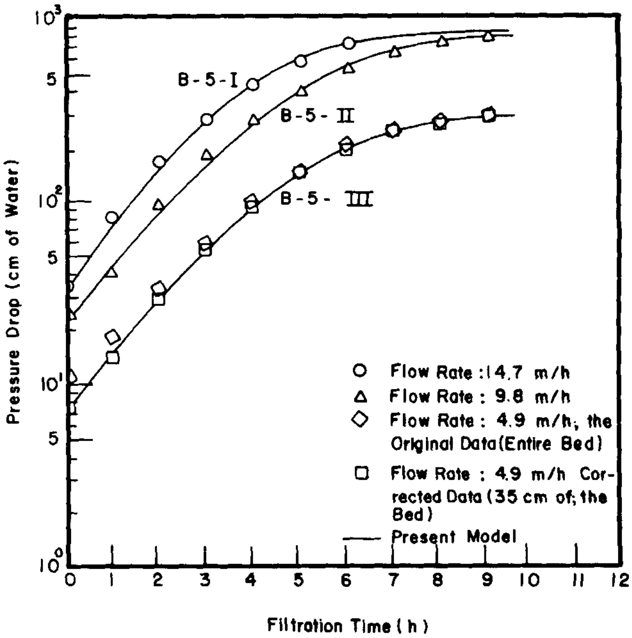

Figure 5. Fitting present model to Huang's (1972) data. Sand size, 0.092 cm; bed depth, 50.8 cm; waste water suspension, 12.5 mg/L.

Runs: B-5-I (α = 0.787 h^{-1}, β = 0.033 h^{-1}, R^2 = 0.996)
B-5-II (α = 0.690 h^{-1}, β = 0.019 h^{-1}, R^2 = 0.997)
B-5-III (α = 0.698 h^{-1}, β = 0.017 h^{-1}, R^2 = 0.999)

linear velocity of the slurry flowing through the pores at the moment t as $v(t)$ and the linear velocity at the onset of the filtration as v_0, we have

$$n_0 \pi r^2 v_0 = \{n_0 - E[N(t)]\}\pi r^2 v(t) \tag{19}$$

or, upon rearrangement,

$$v(t) = \frac{n_0}{n_0 - E[N(t)]} v_0 \tag{20}$$

The pressure drop for laminar flow through a packed bed can be adequately described by the Carman-Kozeny equation (Zenz and Othmer, 1960)

$$-\frac{\Delta P}{L} = 150 \frac{(1-\epsilon)^2}{\epsilon^3} \frac{\mu u}{d_p^2} \tag{21}$$

The superficial velocity, u, in this equation can be expressed in terms of linear velocity, v, as

$$u = \epsilon v \tag{22}$$

Substituting Eq. 22 into Eq. 21 yields

$$-\frac{\Delta P}{L} = 150 \left(\frac{1-\epsilon}{\epsilon}\right)^2 \frac{\mu v}{d_p^2} \tag{23}$$

However, the change of fluid velocity in the pores during a filtration run is expressed by Eq. 20. Thus, substituting Eq. 20 into Eq. 23 results in (Hsu and Fan, 1984)

$$-\frac{\Delta P(t)}{L} = 150 \left(\frac{1-\epsilon}{\epsilon}\right)^2 \frac{\mu v_0}{d_p^2} \left\{\frac{n_0}{n_0 - E[N(t)]}\right\} \tag{24}$$

The expression for $E[N(t)]$, Eq. 17, can now be substituted into Eq. 24 to obtain

$$-\frac{\Delta P(t)}{L} = 150 \left(\frac{1-\epsilon}{\epsilon}\right)^2 \frac{\mu v_0}{d_p^2} \left[\frac{\alpha + \beta}{\beta + \alpha e^{-(\alpha+\beta)t}}\right] \tag{25}$$

The linear velocity, v, and the term involving porosity, $[(1-\epsilon)/\epsilon]^2$, in Eq. 23 are expected to be variable and interrelated. For simplicity, however, we attribute an increase in the pressure drop only to the increase in the linear velocity of flow; this is equivalent to assuming that the porosity term, $[(1-\epsilon)/\epsilon]^2$, in Eq. 25 remains unchanged. Equation 25, therefore, reduces to

$$-\frac{\Delta P(t)}{L} = \left(-\frac{\Delta P}{L}\right)_0 \left[\frac{\alpha + \beta}{\beta + \alpha e^{-(\alpha+\beta)t}}\right] \tag{26}$$

where $(-\Delta P/L)_0$ is a constant and reflects the initial pressure drop through filter.

Note that the parameters α and β in Eq. 26 are assumed to be constant for each filtration run. If the change in suspension concentration is appreciable or the filter bed is of multimedia, it may be desirable to divide the bed along its axis into more than one compartment and to simulate the pressure drop dynamics through each compartment according to Eq. 26 (Fan et al., 1985). Another approach is to resort to expression

$$-\frac{\Delta P(t)}{L} = \left(-\frac{\Delta P}{L}\right)_0 \frac{1}{L} \int_0^L \frac{\alpha(z) + \beta(z)}{\{\alpha(z) + \beta(z) e^{-[\alpha(z)+\beta(z)]t}\}} dz \tag{26a}$$

where the parameters α and β are functions of the axial position, z.

COMPARISON OF THE MODEL WITH THE EXPERIMENTAL DATA AND DISCUSSION

The experimental results by previous researchers (Eliassen, 1935; Ives, 1961; Rimer, 1968; Deb, 1969; Huang, 1972) have been analyzed based on the present model; the results are shown in Figures 1 through 9.

The values of α and β in the model equation, Eq. 26, have been determined by the following procedure (see Appendix C). The preliminary value of α is first determined from the initial slope of

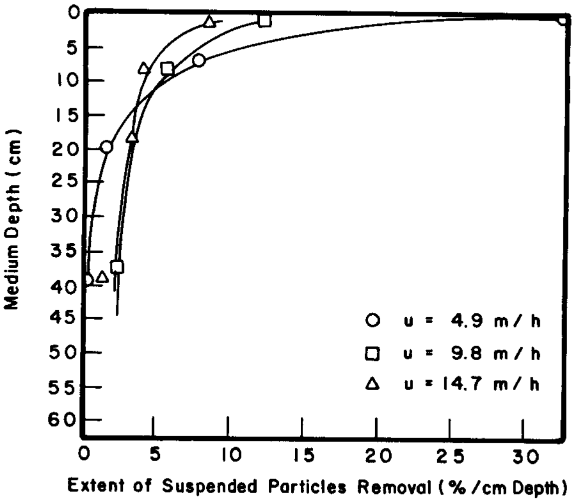

Figure 6. Extent of suspended particles removal per cm of depth in a unisized sand filter in Huang's (1972) data.

Sand size, 0.092 cm; filtration time, 1 h; Runs: B-5-I, -II, -III.

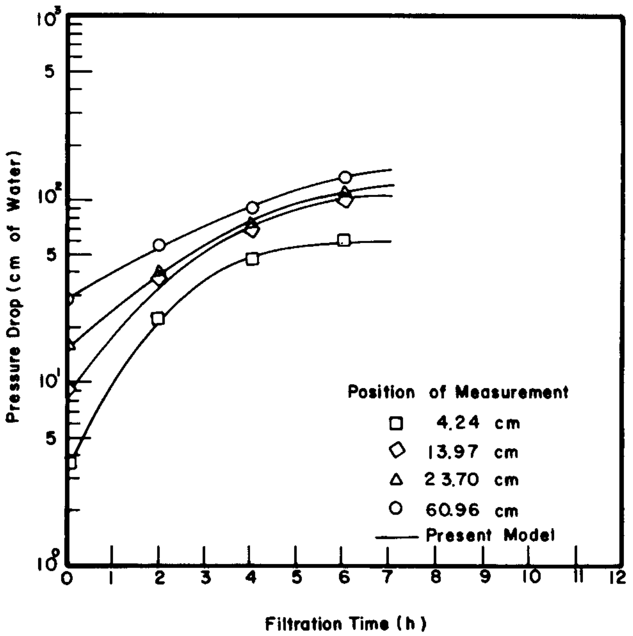

Figure 7. Fitting present model to Rimer's (1968) data. Sand size, 0.046 cm; bed depth, 60.96 cm; flow rate, 7.3 m/h; $FeCl_3$ suspension, 5 mg/L, pH = 8.3.

Run VII-2: Position 4.24 cm (α = 1.035 h^{-1}, β = 0.069 h^{-1}, R^2 = 0.986)
Position 13.47 cm (α = 0.718 h^{-1}, β = 0.066 h^{-1}, R^2 = 0.981)
Position 23.70 cm (α = 0.511 h^{-1}, β = 0.063 h^{-1}, R^2 = 0.997)
Position 60.96 cm (α = 0.347 h^{-1}, β = 0.060 h^{-1}, R^2 = 0.998)

the time-dependent pressure drop curve. With this value in hand the preliminary value of β is estimated from the entire set of the data. Finally, the values of α and β are recovered by using this set of preliminary values as a starting point of search for the minimum of sum of squares residual by means of a nonlinear optimizing technique (Marquardt, 1963). The initial pressure drop per unit depth $(-\Delta P/L)_0$, which is also contained in the equation, can be estimated by the Carman-Kozeny equation or, more accurately, can be measured at the onset of the experimental run. The measured values of the initial pressure drop have been used in the present work.

Deb (1969) conducted his experiment by allowing turbid water to flow through a deep sand bed with uniform grain size of 0.647 mm at a constant filtration rate. The concentration of Fuller's earth particles in turbid water was 45×10^{-4} vol%. The pressure drops were measured at various depths during the filtration run without disturbing the normal operation. The present model, in which the scouring effect is negligible, giving rise to a pure birth process, appears to adequately explain the increase in the pressure drop (see Figure 1). The negligible scouring effect seemed to be due to the short operating time as well as the diluteness of influent concentration, for which the deposit accumulation within the bed was small so that the blockage mechanism became far more dominant than the scouring mechanism.

A rapid sand filter with a depth of 61 cm was used by Eliassen (1935) to remove suspended hydrous ferric oxide floc particles from the water. The pressure drop reading was taken every ten hours; the results are shown in Figure 2. Each experimental run lasted sufficiently long to retain a large amount of deposits in the filter. Under such conditions, a significant scouring effect probably took place.

The present model shows a good fit with the data of Huang (1972), which appear in Figures 3 through 5; the only exception is run B-5-III in Figure 5. This deviation was probably caused by the shallow penetration of suspended particles along the bed. Huang measured the concentration of suspended particles at different depths of the bed under various flow rates as illustrated in Figure 6. Notice that when the flow rate was 4.9 m/h, the filter medium beyond a depth of 35 cm did not contribute to the further removal of suspended particles, rendering the lower 15.8 cm of the bed unutilized. Thus, we can postulate that the present model is only applicable to the first 35 cm of the bed, as illustrated in Figure 5.

The results of Rimer (1968) are shown in Figures 7 and 8, and those of Ives (1961) in Figure 9. The present model agrees well with the data of Rimer from the entire bed depth. However, the model can describe Ives' data only from the first portion of the bed. This is indicative of the shallow penetration discussed in the preceding paragraph. Ives measured in detail the specific deposit as a function of time and bed positions by means of the radioactive algae technique and determined conclusively that the penetration was indeed shallow.

The present model equation, Eq. 26, contains two adjustable, parameters, α and β. These two parameters are functions of many factors, among which are the size distributions of collector grains and suspended particles, properties of the liquid and involved surfaces, filtration rate, bed porosity, and suspension concentration.

The parameters α and β increase with an increase in the filtration rate as illustrated in Figure 4. It is plausible that as the filtration rate increases the amount of the suspended particles entering the filter bed per unit time also increases, thus enhancing the probability of the open pores to be blocked. In other words, the intensity of blockage, λ_n, increases. From Eq. 5 we see that for the same number of open pores, $(N_0 - n)$, the blockage constant, α, increases with the increase in the filtration rate. Furthermore, this increase enhances the scouring capability of the flow which in turn enhances the scouring intensity, μ_n. Equation 6 indicates that for the same amount of trapped particles, n, the scouring constant, β, is magnified as the filtration rate increases.

The parameters are functions of the suspension concentration which decreases along the bed during a filtration run. Thus, these two parameters might also decrease with the bed depth as indicated in Figure 7. In contrast, α decreases monotonically with the bed depth while β varies irregularly, as indicated in Figure 8. Notice that the bed in Figure 7 is of unisized sand while the bed in Figure 8 is of multimedia (anthracite, sand, and garnet). It appears that

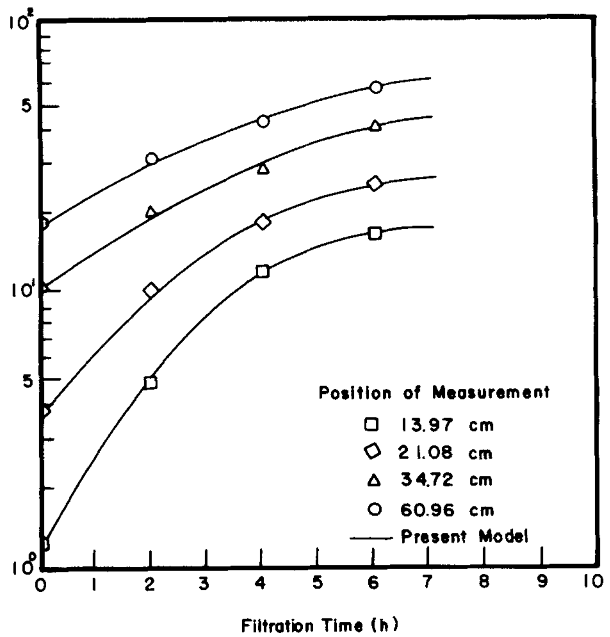

Figure 8. Fitting present model to Rimer's (1968) data. Bed depth, 60.96 cm: anthracite, 20.32 cm, d_a = 0.1 cm; sand, 20.32 cm, d_s = 0.071 cm; garnet 20.32 cm, d_g = 0.059 cm; flow rate, 7.3 m/h.

Run VII-3: Position 13.97 cm (α = 0.778 h^{-1}, β = 0.055 h^{-1}, R^2 = 0.999)
Position 21.08 cm (α = 0.558 h^{-1}, β = 0.081 h^{-1}, R^2 = 0.994)
Position 34.72 cm (α = 0.319 h^{-1}, β = 0.071 h^{-1}, R^2 = 0.986)
Position 60.96 cm (α = 0.276 h^{-1}, β = 0.088 h^{-1}, R^2 = 0.992)

α depends strongly on the suspension concentration while β depends weakly on the suspension concentration; the latter is also highly dependent on the type of bed grains and bed structure, i.e., the spatial distribution of these grains.

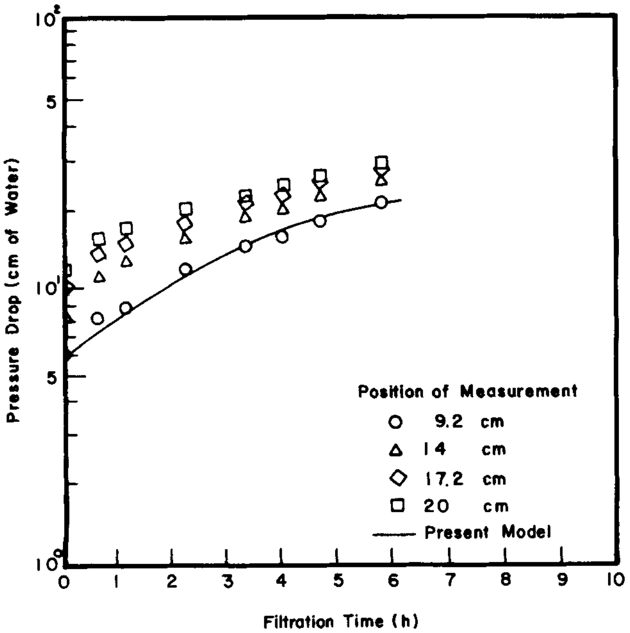

Figure 9. Fitting present model to Ives' (1961) data. Sand size, 0.0544 cm; bed depth, 20 cm; flow rate, 4.9 m/h; chlorella algae suspension, 135 × 10^{-4} vol %.

Run 1B: Position 9.2 cm (α = 0.359 h^{-1}, β = 0.102 h^{-1}, R^2 = 0.986)

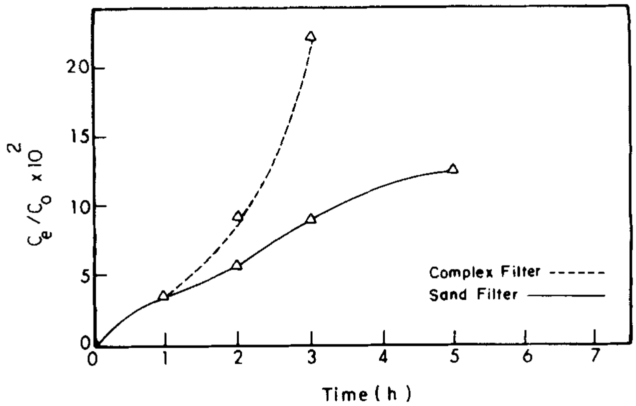

Figure 10. Ratio of effluent to influent ferric iron concentrations, C_e/C_0 as a function of time; Run VII-2 (Rimer, 1968).

The results obtained by Rimer (1968) and Deb (1969) indicate that the operation times of their experimental runs were roughly equal, but the resultant pressure drop dynamics were substantially different. The blockage constant, α, is 0.347 for Rimer's filter, but only 0.130 for Deb's filter, as indicated in Figures 7 and 1, respectively. It appears that the blockage mechanism is more dominant in Rimer's filter than in Deb's filter; also, the scouring mechanism is more dominant in the former than in the latter as discussed previously. Thus, the birth-death model must be employed to depict Rimer's filter while the use of a simple pure birth process is more than adequate for representing Deb's filter within the accuracy of experimental data.

In Rimer's filter, the effluent concentration continued to increase with time, as shown in Figure 10. This is indicative of the scouring phenomenon; recall that the scouring rate increases with an increase in the deposit accumulation. In contrast, Deb's effluent concentration data illustrated in Figure 11 do not exhibit such a monotonically increasing trend. The substantially different pressure drop dynamics between the two filters may be caused by numerous factors, such as the properties of suspension, flow rate, and size of sand grains.

In deriving the present model, we assume that the overall change in the porosity of the filter bed is small enough that the term [(1 −

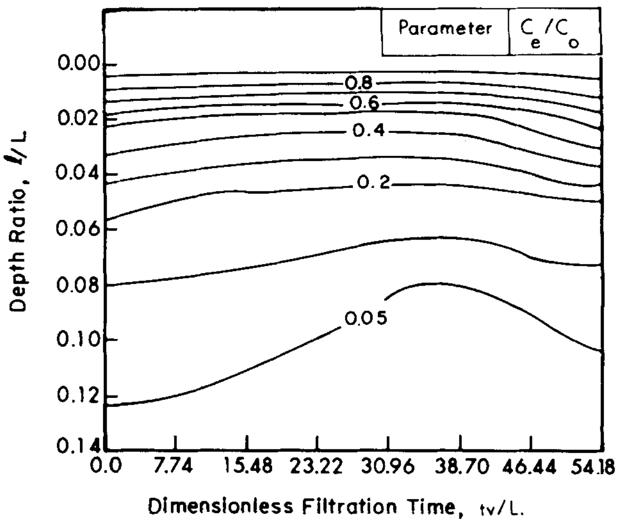

Figure 11. Isoconcentration ratio lines; Run I (Deb, 1969).

$\epsilon)/\epsilon]^2$ in the Carman-Kozeny equation remains essentially constant. This is entirely plausible for a deep bed filtration process which is commonly used for clarification, i.e., for separation of fine particles from very dilute suspensions. For example, Ives (1961) and Deb (1969), measured the specific deposit σ at different depths in their filters. At the end of a filtration run Ives found that the specific deposit at a depth of $0.1L$ (2 cm) was less than 0.02 and essentially zero at a depth of $0.3L$ (6 cm), while Deb found that the specific deposit was approximately 0.02 at a depth of $0.1L$ (6 cm) and 0.01 at a depth of $0.12L$ (7.2 cm) toward the end of a filtration run.

As mentioned in the preceding section, if the change in suspension concentration along the filter bed is appreciable or if the filter bed is of multimedia, it may be desirable to divide the bed along its axis into more than one compartment and to simulate the dynamics of pressure drop through each compartment (or layer) according to Eq. 26. In practice, however, it is more than adequate to consider a deep bed filter as a single compartment and to employ a lumped model to simulate its pressure drop dynamics as shown in Figures 1–5 and in Figures 7–9.

The stochastic birth-death model considered here is assumed to be time-homogeneous, in which both parameters, α and β, are not functions of time because of the constant filtration rate and suspension concentration, i.e., the constant operating conditions. For the case where the filtration rate or suspension concentration varies with time during a filtration run, the birth-death model is still applicable, but both parameters will be functions of time; in other words, the process described by the model is the time-heterogeneous process.

Finally, to gain a deeper understanding of the filtration process, models of various significant physical phenomena taking place in the filter bed, e.g., particle deposition in the filter and the wall-flow interaction, should be incorporated into the present model. Investigations of these phenomena are, however, outside the scope of the present study.

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NOTATION

C_i	= influent slurry concentration
d_a	= diameter of the anthracite
d_p	= average diameter of the grain particles
d_s	= diameter of the sand
$E[N(t)]$	= mean value of the random variable $N(t)$
$G(s,t)$	= probability generating function
L	= depth of the bed
n	= number of pores clogged per unit volume of the bed
n_0	= total number of open pores susceptible to blockage per unit volume of the bed
$N(t)$	= a random variable which describes the number of blocked pores per unit volume of the bed at the moment t
$p_n(t)$	= probability that exactly n pores are blocked at the moment t
$-\Delta P/L$	= pressure drop per unit length of the filter
R^2	= fraction of variation
r	= radius of pore
s	= variable of the probability generating function
t	= time
u	= superficial velocity
v	= linear velocity
$\mathrm{Var}[N(t)]$	= variance of the random variable $N(t)$

Greek Letters

λ_n	= intensity of the birth transition
μ_n	= intensity of the death transition
α	= proportionality constant defined in Eq. 5, the blockage constant
β	= proportionality constant defined in Eq. 6, the scouring constant
σ	= specific deposit, volume of deposit per unit volume of the filter bed

APPENDIX A

An Alternative Derivation of the Master Equations

The master equations, Eqs. 3 and 4, in the text can also be derived through the consideration of a time-homogeneous Markov process. Let $p_{ij}(\tau,t)$ denote the transition probability that the system in the state of i pores blocked at time τ will change to the state of j pores blocked at time t; $i,j = 0,1,2,\ldots,n_0$. It is assumed that within the small time interval $(t,t+\Delta t)$

$$p_{ij}(t,t+\Delta t) = v_{ij}\Delta t + o(\Delta t), \quad i \neq j;\, i,j = 0,1,2,\ldots,n_0 \tag{A1}$$

$$1 - p_{ii}(t,t+\Delta t) = -v_{ii}\Delta t + o(\Delta t), \quad i = 0,1,2,\ldots,n_0 \tag{A2}$$

where v_{ij} and v_{ii} are the intensity functions.

It can be shown that the transition probabilities, $\{p_{ij}(\tau,t)\}$ satisfy the following Kolmogorov forward differential equations (Chiang, 1980),

$$\frac{d}{dt}p_{ij}(\tau,t) = \sum_{k=0}^{n_0} p_{ik}(\tau,t)v_{kj}, \quad i,j = 0,1,2,\ldots,n_0, \tag{A3}$$

with the initial condition

$$p_{ij}(\tau,\tau) = \begin{cases}1, & i = j\\ 0, & i \neq j\end{cases} \tag{A4}$$

Let $p_{ij}(t)$ denote the transition probability from time 0 to time t. In matrix notation, Eqs. A3 and A4 can be written, respectively, as

$$\frac{d}{dt}\mathbf{P}(t) = \mathbf{P}(t)\mathbf{U} \tag{A5}$$

and

$$\mathbf{P}(0) = \mathbf{I} = \text{identity matrix} \tag{A6}$$

where

$$\mathbf{P}(t) = \begin{bmatrix} p_{00}(t) & p_{01}(t) & \cdots & p_{0n_0}(t)\\ p_{10}(t) & p_{11}(t) & \cdots & p_{1n_0}(t)\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ p_{n_00}(t) & p_{n_01}(t) & \cdots & p_{n_0n_0}(t)\end{bmatrix} \tag{A7}$$

and

$$\mathbf{U} = \begin{bmatrix} v_{00} & v_{01} & \cdots & v_{0n_0}\\ v_{10} & v_{11} & \cdots & v_{1n_0}\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ v_{n_00} & v_{n_01} & \cdots & v_{n_0n_0}\end{bmatrix} \tag{A8}$$

In the present model, it is assumed that given $N(t) = n$:

1. The transition probabilitty that more than one pore will be blocked or scoured is $o(\Delta t)$, i.e.,

$$v_{ij} = 0, \quad |j - i| \geq 2; \quad i,j = 0,1,\ldots,n_0. \tag{A9}$$

2. The transition probability that an open pore will be blocked during the interval $(t, t + \Delta t)$ is $\lambda_n \Delta t + o(\Delta t)$, i.e.,

$$v_{ij} = \lambda_n, \quad i = n; \quad n = 0,1,2,\ldots,n_0 - 1 \qquad j = n + 1 \tag{A10}$$

3. The transition probability that a blocked pore will be scoured during the interval $(t, t + \Delta t)$ is $\mu_n \Delta t + o(\Delta t)$, i.e.,

$$v_{ij} = \mu_n, \quad i = n; \quad n = 1,2,\ldots,n_0; \qquad j = n - 1 \tag{A11}$$

4. The transition probability of no change in the interval $(t, t + \Delta t)$ is $[1 - \lambda_n \Delta t - \mu_n \Delta t - o(\Delta t)]$, i.e.,

$$v_{ii} = -(\lambda_n + \mu_n), \quad i = n; \quad n = 0,1,2,\ldots,n_0 \tag{A12}$$

Then Eq. A8 reduces to

$$\mathbf{U} = \begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & \cdots & 0 \\ \mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & 0 & \cdots & 0 \\ 0 & \mu_2 & -(\lambda_2 + \mu_2) & \lambda_2 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & -\mu_{n_0} \end{bmatrix} \tag{A13}$$

Let $p_n(t)$ be the probability that exactly n pores are blocked at time t, and $\mathbf{P}^*(t)$ the vector of probabilities, i.e.,

$$\mathbf{P}^*(t) = p_0(t) p_1(t) p_2(t) \ldots p_{n_0}(t) \tag{A14}$$

It can be seen that

$$\mathbf{P}^*(t) = \mathbf{P}^*(0)\mathbf{P}(t) \tag{A15}$$

Multiplying both sides of Eq. A5 with $\mathbf{P}(0)$, we obtain

$$\frac{d}{dt} \mathbf{P}^*(0)\mathbf{P}(t) = \mathbf{P}^*(0)\mathbf{P}(t)\mathbf{U} \tag{A16}$$

or

$$\frac{d}{dt} \mathbf{P}^*(t) = \mathbf{P}^*(t)\mathbf{U} \tag{A16a}$$

Expanding Eq. A16a, we obtain

$$\frac{dp_n(t)}{dt} = \lambda_{n-1} p_{n-1}(t) - (\lambda_n + \mu_n) p_n(t) + \mu_{n+1} p_{n+1}(t), \quad n \geq 1 \tag{A17}$$

and

$$\frac{dp_0(t)}{dt} = -\lambda_0 p_0(t) + \mu_1 p_1(t) \tag{A18}$$

which are the same as Eqs. 3 and 4, in the text.

APPENDIX B

Solution of the Probability Generating Function of the Master Equations

The system of differential equations, Eqs. 7 and 8, in the text, describe the linear stochastic birth-death process as

$$\frac{dp_n(t)}{dt} = \alpha[n_0 - (n - 1)] p_{n-1}(t) - [\alpha(n_0 - n) + \beta n] p_n(t) + \beta(n + 1) p_{n+1}(t), \quad n \geq 1 \tag{B1}$$

and

$$\frac{dp_0(t)}{dt} = -\alpha n_0 p_0(t) + \beta p_1(t) \tag{B2}$$

The mean number of the blocked pores at a given moment t is to be determined; it is

$$E[N(t)] = \sum_{n=0}^{n_0} n p_n(t) \tag{B3}$$

To evaluate this value, we use the method of probability generating function which is defined as

$$G(s,t) = \sum_{n=0}^{n_0} s^n p_n(t) \tag{B4}$$

Multiplying both sides of Eq. B1 by the respective s^n's and Eq. B2 by s^0, the following expression is obtained:

$$\sum_{n=0}^{n_0} s^n \frac{dp_n(t)}{dt} = [\beta + (\alpha - \beta)s - \alpha s^2] \left[\sum_{n=0}^{n_0} n s^{n-1} p_n(t)\right] + [\alpha n_0 (s - 1)] \left[\sum_{n=0}^{n_0} s^n p_n(t)\right] \tag{B5}$$

Since

$$\frac{\partial G(s,t)}{\partial t} = \sum_{n=0}^{n} s^n \frac{dp_n(t)}{dt} \tag{B6}$$

and

$$\frac{\partial G(s,t)}{\partial s} = \sum_{n=0}^{n_0} n s^{n-1} p_n(t), \tag{B7}$$

Equation B5 reduces to

$$\frac{\partial G(s,t)}{\partial t} = \frac{\partial G(s,t)}{\partial s} [\beta + (\alpha - \beta)s - \alpha s^2] + G(s,t)[\alpha n_0 (s - 1)] \tag{B8}$$

The initial and boundary conditions can be expressed as

$$G(s,0) = 1 \tag{B9}$$

$$G(1,t) = 1 \tag{B10}$$

Assume that the solution of Eq. B8 take the form of

$$G(s,t) = [f(s) + g(s,t)]^{n_0} \tag{B11}$$

Substituting this into Eq. B8 yields

$$\frac{\partial g(s,t)}{\partial t} = [\beta + (\alpha - \beta)s - \alpha s^2] \left[\frac{df(s)}{ds} + \frac{\partial g(s,t)}{\partial s}\right] + [\alpha(s - 1)][f(s) + g(s,t)] \tag{B12}$$

TABLE C-1. COMPARISON OF HUANG'S DATA (RUN B-5-II, 1972) AND RESULTS FROM THE PRESENT MODEL

Time h	$(-\Delta P)_{exp}$ cm H_2O	$(-\Delta P)_{cal}$ cm H_2O
0	21.67	21.67
1	40.63	42.85
2	95.92	82.55
3	178.6	151.7
4	276.1	258.3
5	390.1	394.8
6	520.2	533.4
7	634.0	644.9
8	733.4	718.8
9	772.1	761.8

Let $f(s)$ and $g(s,t)$ satisfy

$$[\beta + (\alpha - \beta)s - \alpha s^2]\frac{df}{ds} + [\alpha(s-1)]f(s) = 0 \quad \text{(B13)}$$

and

$$\frac{\partial g(s,t)}{\partial t} = [\beta + (\alpha - \beta)s - \alpha s^2]\frac{\partial g(s,t)}{\partial s} + [\alpha(s-1)]g(s,t) \quad \text{(B14)}$$

From Eqs. B9 and B11, we obtain

$$g(s,0) = 1 - f(s) \quad \text{(B15)}$$

From Eqs. B10 and B11, we have

$$f(1) = 1 \quad \text{(B16)}$$

$$g(1,t) = 0 \quad \text{(B17)}$$

Subject to these boundary and initial conditions, Eqs. B13 and B14 are solved, respectively, as

$$f(s) = \frac{\alpha s + \beta}{\alpha + \beta} \quad \text{(B18)}$$

$$g(s,t) = \frac{-\alpha(s-1)e^{-(\alpha+\beta)t}}{\alpha + \beta} \quad \text{(B19)}$$

Therefore, the solution of Eq. B18 becomes

$$G(s,t) = \left[\frac{(\alpha s + \beta) - \alpha(s-1)e^{-(\alpha+\beta)t}}{\alpha + \beta}\right]^{n_0} \quad \text{(B20)}$$

which is the same as Eq. 15 in the text.

APPENDIX C

Estimation of Parameters from the Experimental Data

The values of the parameters α and β in the governing equation of the model (Eq. 26 in the text) have been estimated by minimizing the sum of the squares of deviations between the experimental data and the values obtained from the present model with respect to these parameters. The procedure is illustrated with selected experimental data (Run B-5-II of Huang, 1972), which are reproduced in Table C-1. The procedure is as follows.

1. The preliminary value of α is tentatively estimated from the initial slope of the time-dependent pressure drop curve as

$$\alpha = \frac{\ln(40.63) - \ln(21.67)}{1.0 - 0.0} \approx 0.628$$

2. With the tentative value of α in hand, the value of β is tentatively estimated as 0.0154 from the entire set of the data by minimizing the sum of the squares of deviations between the experimental data and values obtained from the present model with respect to β.

3. The set of α and β, which minimizes the sum of the squares of deviations, has been obtained by means of the Marquardt (1963) least-squares estimation technique, with the point corresponding to the tentative values of α and β obtained in the first two steps as the starting point. The results are shown in Tables C-1 and C-2. The R^2 value has been calculated as

$$R^2 = 1.0 - \frac{\text{Residual}}{\text{Corrected Total}} = 1.0 - \frac{1{,}855.58}{736{,}585.72} \approx 0.997$$

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TABLE C-2. NONLINEAR LEAST-SQUARES SUMMARY STATISTICS

Sources	d.f.	Sum of Squares	Mean Squares	Parameters	Estimate	Asymptotic Std. Error	Asymptotic 95% Confidence Interval Lower	Upper
Regression	2	2,076,296.57	1,038,148.58	α	0.690	0.01054	0.6659	0.7143
Residual	8	1,855.58	231.95	β	0.019	0.00063	0.0175	0.0205
Uncorrected Total	10	2,078,152.15						
Corrected Total	9	736,585.72						

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